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The equilibrium and stability of a translating cavity in a liquid

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A strong sound field in a liquid may generate small cavities that move rapidly through the liquid. This paper is an analysis of the equilibrium states, and the conditions for their stability, of a cavity in translational motion through an inviscid, incompressible liquid. A principal conclusion is that translating cavities have a stable size and shape for a wide range of conditions. The equilibrium shape of a stable cavity is approximately an oblate spheroid. It is also found that the presence of translational motion contributes an outward dynamic pressure that tends to enlarge the cavity. As a result, the equilibrium radius of a translating cavity is greater than the equilibrium radius of the same cavity at rest.

1. Introduction

A strong sound field in a liquid may generate small cavities that dart about in rapid motion through the liquid. In this paper a study is made of the equilibrium size and shape and the stability of a cavity in translational motion. A principal conclusion is that translating cavities have a stable size and shape for a wide range of conditions.

A translating cavity will be shown to differ from a stationary cavity primarily in two respects: the shape of the cavity is deformed, and the average pressure in the liquid surrounding the cavity is less than the hydrostatic pressure at infinity. This reduction of pressure tends to enlarge the cavity. As a result, the equilibrium radius of a translating cavity is greater than the equilibrium radius of the same cavity at rest.

The motion of a body through a fluid differs from the motion of the same body in free space in one important respect. When a body moves through the fluid, the fluid in the neighbourhood of the body must also move; this fluid motion is directly related to the motion of the body. Therefore, dynamic quantities such as kinetic energy and momentum consist of two contributions, the kinetic energy and momentum of the body itself, and the kinetic energy and momentum of the fluid flow associated with the motion of the body. It is often convenient to account for the dynamic contribution of the fluid by attributing to the body in motion a total effective mass which is the sum of the intrinsic mass of the body itself and an induced mass arising through the motion of the fluid. For example, a sphere

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of mass m, volume V and density $\rho' = m/V$, moving through an incompressible liquid of density ρ, has an induced mass of $\frac{1}{2}\rho V$ and, therefore, a total effective mass $M = m + \frac{1}{2}\rho V = (\rho' + \frac{1}{2}\rho)V$. The concept of induced mass is discussed in greater detail by Birkhoff (1950).

The motion of cavities through a liquid represents the special case in which the intrinsic mass (in this case the mass of whatever gas may be present in the cavity) is negligible, and the total mass can be assumed to be given by the induced mass alone. By means of this concept of induced mass it is possible to speak of such dynamic quantities as the kinetic energy and momentum of a cavity, in spite of the fact that a cavity has negligible intrinsic mass. Thus, an expression like the 'momentum of a cavity' is an abbreviated way of referring to the momentum of the fluid in the neighbourhood of the cavity.

Translation of cavities in an intense sound field has been described by several authors. At times the velocity of translation is quite large. For example, Willard (1953) observed cavities moving at velocities up to 1000 cm/sec through the focal region of a focused travelling wave, and Barger (1964) recently observed a cavity moving outward from the pressure antinode at the centre of a spherical standing wave at a velocity of 125 cm/sec.

Much of the present knowledge of translating cavities comes not from acoustic cavitation but, rather, from studies of gas bubbles rising through a liquid. Haberman & Morton (1953) report that rising bubbles have one of three characteristic shapes. Very small bubbles remain spherical; bubbles of intermediate size are deformed into an oblate spheroid; and large bubbles form what are known as spherical cap bubbles. Hartunian & Sears (1957) observed that the very small bubbles rise in a straight, vertical path. In liquids of low viscosity, however, bubbles exceeding a well-defined critical size for each liquid have an unstable trajectory, and the rising bubble will oscillate or spiral about the vertical direction. The shape of these bubbles is an oblate spheroid. The significant physical quantity that determines the shape and path of bubbles in relatively inviscid liquids is the Weber number, a dimensionless parameter given by $\rho u^2 R/\sigma$.

The problem of translational motion of bubbles has arisen also in studies of underwater explosion bubbles. An approximate theory of the dynamics of explosion bubbles was worked out by Taylor (1963, pp. 320–36, 337–53), who, in order to avoid the complexities resulting from shape distortion, assumed the bubble shape to be a sphere. This spherical model predicts that the presence of translational motion tends to cushion or oppose collapse of the bubble. In fact, according to the spherical model, an empty cavity cannot contract to zero radius as long as there is translational motion. Approximate calculations by Taylor indicate that the presence of translational motion can increase the minimum radius attained by a pulsating cavity by a factor of 3·3 and decrease by a factor of 100 the maximum gas pressure attained during contraction. For this same example Taylor found that the amount of energy radiated as a pressure wave during cavity rebound is reduced to one-tenth the amount of energy that would be radiated if there were no translational motion.

The present paper consists of an analysis of the equilibrium size and shape of a cavity in uniform translational motion through an inviscid liquid. This task will be accomplished by first deriving general equations of motion for the size, shape and translational velocity of the cavity. From these equations conditions will then be determined for the equilibrium state of a translating cavity and for the stability of the cavity size and shape. The stability of the trajectory, which was examined by Hartunian & Sears, will not be considered here. A more detailed and extensive discussion of the dynamics of translating cavities is available in a recent report (Eller 1966).

2. Equations of motion for a translating cavity

The analysis is begun by considering a cavity with some given initial momentum, translating in a straight line through an infinite, incompressible and inviscid liquid. The hydrostatic pressure P_0 is considered to be constant in time and uniform in space throughout the liquid, and no other external forces exist.

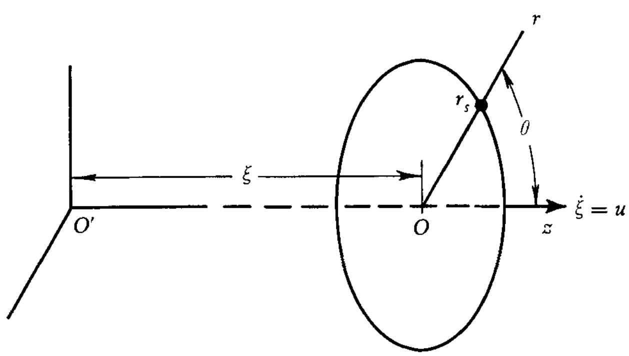

FIGURE 1. Co-ordinate system for a translating cavity.

It may be assumed that at some previous time an external force, due either to gravity or a sound field, did exist and gave the cavity its initial momentum. The present analysis begins when the cavity enters a region in which no such external forces exist, and the particular means by which the cavity acquired its initial momentum are not considered. It will be seen that, because both viscous forces and external forces are neglected, the translational momentum is a constant.

The surface of the cavity is treated as a free boundary whose shape and position are to be determined; it is noted that the shape is not necessarily a sphere. However, it is assumed that the shape of the cavity is symmetric about the direction of translation. The cavity may contain gas, but the density of the gas is much less than the density of the liquid so that the kinetic energy of the gas may be neglected. The surface tension of the liquid is included.

We shall assume for the present that the position of the cavity surface and the translational velocity are functions of time. The position of the cavity surface is given by the function $r = r_s(\theta, t)$, or $F(r, \theta, t) \equiv r - r_s = 0$, where r and θ are spherical co-ordinates in a co-ordinate system whose origin O is located at the centre of the cavity, a distance $\xi(t)$ from some fixed reference point O'. The centre of the cavity moves in the z direction with a velocity $\dot{\xi} = u(t)$ with respect to the rest frame, as indicated in figure 1. The co-ordinate system (O, r, θ) attached to the centre of the cavity is also moving with velocity $u(t)$. At first it may appear that the problem is treated in an accelerating co-ordinate system that would require the inclusion of new forces. However, the moving co-ordinate system is used only to describe the cavity shape; the dynamics of the motion will be described with respect to the inertial, fixed co-ordinate system. Thus, in the expression $\frac{1}{2}\rho v^2$ for the kinetic energy of a fluid particle, the velocity $\mathbf{v}$ at a point in the liquid is measured not with respect to the moving frame, but in the rest frame. Similarly, the negative gradient of the velocity potential ϕ gives the fluid velocity in the rest frame.

The function r_s may be expanded in a series of Legendre polynomials as

$$r_s(\theta, t) = R(t) + \sum_{n=2}^{\infty} c_n(t)\, P_n(\cos\theta).$$

In this expression R is the coefficient of the zero-order Legendre polynomial $P_0 = 1$. A term $c_1 P_1$ is identically zero by the choice of origin and does not appear. Such a term would describe translation of the cavity with respect to a fixed origin; this motion is accounted for here by allowing the origin of the co-ordinate system to move with the centre of the cavity. It is convenient to write this expansion as

$$r_s(\theta, t) = R(t)\left[1 + \sum_{n=2}^{\infty} x_n(t)\, P_n(\cos\theta)\right] \equiv R(t) f(\theta, t). \tag{1}$$

In the following analysis we shall first solve Laplace's equation for the velocity potential ϕ. Next, the kinetic energy T and the potential energy U will be expressed as functions of R, $\dot{R}$, x_n, $\dot{x}_n$ and the translational velocity u. Finally, the Lagrangian of the system is given by $L = T - U$, and equations of motion for the cavity are found by applying Lagrange's equations of motion to this Lagrangian.

Solution of Laplace's equation

The motion of the incompressible liquid is assumed to be irrotational and, therefore, may be described by a velocity potential such that $-\nabla\phi = \mathbf{v}$. The potential ϕ is a solution of Laplace's equation that vanishes at infinity and is represented by the series

$$\phi = \sum_{n=0}^{\infty} R^{n+2}(t)\, b_n(t)\, \frac{P_n(\cos\theta)}{r^{n+1}}. \tag{2}$$

The time-dependent coefficients $R^{n+2}(t)\, b_n(t)$ have been written in this factored form for the sake of convenience. The problem here is to evaluate the coefficients b_n as functions of R, $\dot{R}$, x_n, $\dot{x}_n$ and u. This goal may be accomplished once the normal velocity at the cavity boundary is specified.

The normal velocity of a fluid particle at the surface is the sum of the normal velocity in the moving co-ordinate system with origin at the centre of the cavity, plus a term due to the translation of the cavity with velocity u. The normal velocity in the moving system is $(\partial r_s/\partial t)_\theta\, \mathbf{i}_r . \mathbf{n}$ where $(\partial r_s/\partial t)_\theta$ is the velocity of the surface in the radial direction at some fixed angle θ. From (1) one has $(\partial r_s/\partial t)_\theta = \dot{R}f + R\dot{f}$, where $\dot{f}$ indicates the time derivative of f with θ fixed and is given by $\dot{f} = \Sigma \dot{x}_n P_n$. The contribution of translation to the normal velocity is $u\mathbf{i}_z . \mathbf{n}$. Therefore, the normal velocity at the surface is

$$\mathbf{v}.\mathbf{n} = -\nabla\phi . \mathbf{n} = (\dot{R}f + R\dot{f})\,\mathbf{i}_r . \mathbf{n} + u\mathbf{i}_z . \mathbf{n}. \tag{3}$$

The unit normal vector to the surface $F(r, \theta, t) = 0$, pointing outward from the centre of the cavity, is

$$\mathbf{n} = \frac{\nabla F}{|\nabla F|} = \frac{\mathbf{i}_r - (1/f)(\partial f/\partial \theta)\,\mathbf{i}_\theta}{[1 + (1/f^2)(\partial f/\partial\theta)^2]^{\frac{1}{2}}}. \tag{4}$$

The quantity $\nabla\phi$, appearing in (3), may be calculated from (2). Then, with the relations

$$\mathbf{i}_z . \mathbf{i}_r = \cos\theta = P_1 \quad \text{and} \quad \mathbf{i}_z . \mathbf{i}_\theta = -\sin\theta = \partial P_1/\partial\theta,$$

equations (3) and (4) yield

$$\sum_{n=0}^{\infty} \frac{(n+1)b_n P_n}{f^{n+2}} + \sum_{n=1}^{\infty} \frac{b_n}{f^{n+3}} \frac{\partial P_n}{\partial\theta} \frac{\partial f}{\partial\theta} = \dot{R}f + R\dot{f} + uP_1 - \frac{u}{f}\frac{\partial f}{\partial\theta}\frac{\partial P_1}{\partial\theta}. \tag{5}$$

Equation (5) determines the coefficients b_n as functions of the variables R, $\dot{R}$, x_n, $\dot{x}_n$ and u. It is seen from (5) and (2) that b_n and ϕ are linear functions in $\dot{R}$, u and $\dot{x}_n$, though not in R and x_n, and may be written in the general form

$$b_n = \dot{R}b_{n0} + ub_{n1} + \sum_{2}^{\infty} \dot{x}_i b_{ni} = \sum_{0}^{\infty} \dot{q}_i b_{ni}, \tag{6}$$

$$\phi = \dot{R}\phi_0 + u\phi_1 + \sum_{2}^{\infty} \dot{x}_i \phi_i = \sum_{0}^{\infty} \dot{q}_i \phi_i, \tag{7}$$

where b_{ni} and ϕ_i are functions of R and x_n. The general symbol $\dot{q}$ represents any of $\dot{R}$, u or $\dot{x}_n$ according to the correspondence $\dot{q}_0 = \dot{R}$, $\dot{q}_1 = u$ and $\dot{q}_n = \dot{x}_n$ $(n \geqslant 2)$. It is used to give the co-ordinates a uniform notation and does not represent a new choice of co-ordinates. The term ϕ_i refers to the contribution of $\dot{q}_i$ to the total potential ϕ. For example, $\dot{q}_1\phi_1 \equiv u\phi_1$ is the contribution of translation to ϕ. From (2), (6) and (7), ϕ_i is given by

$$\phi_i = \sum_{0}^{\infty} b_{ni} R^{n+2} P_n \Big/ r^{n+1}. \tag{8}$$

The coefficient b_{ni} refers to the contribution of $\dot{q}_i$ to the nth multipole of ϕ. For example, $b_{11}(R^3P_1/r^2)\,u$ is the contribution of translation to the dipole field.

The b_{ni} coefficients can be found from (5) and (6) as a power series in all of the x_n by means of an iteration procedure. The validity of this procedure requires all x_n to be small enough that the inequality $(r_s - R)/R < 1$ be satisfied for all values of the angle θ. It is convenient, then, for book-keeping purposes to assume that the x_n are small quantities of order of magnitude ϵ, written $x_n = O(\epsilon)$. Products of the form $x_n x_k$ will then be of order ϵ^2, and so on. The b_{ni} coefficients are constructed by first finding the terms in b_{ni} of zero order in ϵ. This result is then used to find the terms of order ϵ. Higher-order terms are found in a similar way. Details and results of this calculation are given by Eller (1966). In the present paper explicit results are needed only for b_{11} and b_{31}, which are given below through the number of terms needed in later calculations:

$$b_{11} = \tfrac{1}{2} - \tfrac{3}{10}x_2 + \frac{9}{4}\sum_{n=2}^{\infty}\left[\frac{(n+1)^2(n+2)}{(2n+3)} + \frac{(n-1)^2 n}{(2n-1)}\right]\frac{x_n^2}{(2n+1)^2}$$
$$-\frac{9}{2}\sum_{n=3}^{\infty}\frac{n^2(n+1)}{(2n-1)(2n+1)(2n+3)}x_{n-1}x_{n+1} + O(\epsilon^3)$$
$$= \tfrac{1}{2} - \tfrac{3}{10}x_2 + 0{\cdot}5228x_2^2 + 0{\cdot}5184x_3^2 + 0{\cdot}5216x_4^2 - 0{\cdot}5143x_2x_4 + \ldots$$
$$+ 0{\cdot}1091x_2^3 + 0{\cdot}2150x_2x_3^2 + \ldots, \tag{9}$$

$$b_{31} = \tfrac{9}{10}x_2 - \tfrac{1}{2}x_4 + O(\epsilon^2). \tag{10}$$

The solution of Laplace's equation is now expressed formally as $\phi = \Sigma \dot{q}_i \phi_i$ with ϕ_i given by (8). This solution is used next to obtain an expression for the kinetic energy that results from the motion of the cavity.

Kinetic energy

The kinetic energy of the system is given by

$$T = \tfrac{1}{2}\rho\int_{\Omega} v^2\, d\Omega = -\tfrac{1}{2}\rho\int_S \phi\,\nabla\phi\,.\,\mathbf{n}\,dS,$$

where ρ is the density of the liquid, $d\Omega$ is the volume element in the liquid, and dS is the element of area on the cavity surface. A similar surface integral over a surface at infinity has no contribution because the integrand $\phi\nabla\phi$ behaves as r^{-3} at infinity, and the integral vanishes. Using (7), one may write

$$T = \frac{1}{2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} M_{ij}\dot{q}_i\dot{q}_j, \tag{11}$$

where
$$M_{ij} = -\rho\int_S \phi_i\nabla\phi_j\,.\,\mathbf{n}\,dS. \tag{12}$$

The kinetic energy is seen to be a quadratic function of the rates of change $\dot{q}_i$. The quantities M_{ij} will be called generalized masses.

In the following analysis an explicit value is needed only for the generalized mass M_{11}, which is the induced mass for translation of a cavity with arbitrary but fixed values of R and x_n. M_{11} is given by

$$M_{11} = -\rho\int \phi_1\nabla\phi_1\,.\,\mathbf{n}\,dS.$$

It can be shown that M_{11} may be written as

$$M_{11} = 4\pi\rho R^3 b_{11} - \rho V. \tag{13}$$

V is the volume of the cavity and is given by

$$V = 2\pi\int_0^{\pi}\int_0^{r_s} r^2\,dr\sin\theta\,d\theta = \tfrac{4}{3}\pi R^3\frac{1}{2}\int_{-1}^{1} f^3\,d\mu,$$

where $\mu = \cos\theta$.

Equation (13) is a familiar result (Taylor 1928) for the induced mass of translation of a body of fixed shape moving through a perfect liquid and holds both for the motion of a solid body as well as for the motion of a cavity. The difference between these two cases is that the total mass of a solid consists of this induced mass plus the intrinsic mass, while the intrinsic mass of the cavity may be neglected.

It is convenient to define the ratios

$$w = \frac{V}{\frac{4}{3}\pi R^3} = \frac{1}{2}\int_{-1}^{1} f^3 d\mu \tag{14}$$

and

$$m = \frac{M_{11}}{\frac{2}{3}\pi\rho R^3} = 6b_{11} - 2w. \tag{15}$$

The volume ratio w is the ratio of the cavity volume to the volume of a sphere with radius R, and the quantity m is the ratio of the induced mass of translation of the cavity to that of a sphere with radius R. The ratios w and m are functions only of the cavity shape, that is, functions of x_n but not of R, and can be expanded as

$$\begin{aligned} w &= 1 + 3\sum_{n=2}^{\infty} \frac{x_n^2}{2n+1} + O(\epsilon^3) \\ &= 1{\cdot}0000 + 0{\cdot}6000x_2^2 + 0{\cdot}4286x_3^2 + 0{\cdot}3333x_4^2 + \ldots + 0{\cdot}0571x_2^3 \\ &\qquad + 0{\cdot}1143x_2x_3^2 + 0{\cdot}1714x_2^2x_4 + 0{\cdot}0779x_3^2x_4 + \ldots, \end{aligned} \tag{16}$$

and

$$\begin{aligned} m &= 1 - \tfrac{9}{5}x_2 + \sum_{n=2}^{\infty}\left[\frac{27(n+1)^2(n+2)}{2(2n+3)(2n+1)^2} + \frac{27(n-1)^2 n}{2(2n-1)(2n+1)^2} - \frac{6}{2n+1}\right]x_n^2 \\ &\qquad - 27\sum_{n=3}^{\infty}\frac{n^2(n+1)}{(2n+1)(2n-1)(2n+3)}x_{n-1}x_{n+1} + O(\epsilon^3) \\ &= 1{\cdot}0000 - 1{\cdot}8000x_2 + 1{\cdot}9371x_2^2 - 3{\cdot}0857x_2x_4 + 2{\cdot}4632x_4^2 + 2{\cdot}2531x_3^2 + \ldots \\ &\qquad + 0{\cdot}5406x_2^3 + 1{\cdot}0615x_2x_3^2 + \ldots. \end{aligned} \tag{17}$$

Potential energy

The potential energy of the system is given by

$$U = \sigma A + P_0 V - \int_{V_0}^{V} p_g(V)\,dV, \tag{18}$$

where σ is the surface tension of the liquid, A is the cavity surface area, P_0 is the hydrostatic pressure in the liquid far from the cavity, and p_g is the pressure of the gas in the cavity. The first two terms represent the work needed to introduce a cavity of volume V and surface area A into the liquid. The third term represents the work to fill the cavity with gas at pressure p_g and volume V from some reference volume V_0. It is assumed that the gas pressure is a function only of the volume of the cavity. The area is given by

$$\begin{aligned} A &= 2\pi\int_{-1}^{1}\left[1 + \frac{1}{r_s^2}\left(\frac{\partial r_s}{\partial\theta}\right)^2\right]^{\frac{1}{2}} r_s^2\, d\mu \\ &= 2\pi R^2\int_{-1}^{1}\left[1 + \frac{1}{f^2}\left(\frac{\partial f}{\partial\theta}\right)^2\right]^{\frac{1}{2}} f^2\, d\mu. \end{aligned}$$

It is convenient to define the ratio

$$s \equiv \frac{A}{4\pi R^2} = \frac{1}{2}\int_{-1}^{1}\left[1+\frac{1}{f^2}\left(\frac{\partial f}{\partial \theta}\right)^2\right]^{\frac{1}{2}} f^2 d\mu, \tag{19}$$

which is the ratio of the cavity surface area to that of a sphere with radius R. s is calculated by expanding the radical in the integrand of (19) in a Taylor series and then integrating to give

$$\begin{aligned} s &= 1+\sum_{n=2}^{\infty}\frac{2+n(n+1)}{2(2n+1)}x_n^2+O(\epsilon^4) \\ &= 1{\cdot}0000+0{\cdot}8000x_2^2+1{\cdot}0000x_3^2+1{\cdot}2222x_4^2+\ldots \\ &\qquad -0{\cdot}2571x_2^4-1{\cdot}6831x_2^2x_3^2-\ldots. \end{aligned} \tag{20}$$

The three ratios m, s and w are important variables used throughout the analysis. They serve as a measure of the departure of a deformed cavity from a sphere, and each ratio has the value unity when the cavity shape is a sphere.

The equations of motion

The kinetic and potential energies have now been expressed as functions of R, $\dot{R}$, x_n, $\dot{x}_n$ and u. The variables R, x_n and ξ, the position of the centre of the cavity, thus serve as generalized co-ordinates for this problem, and their time derivatives $\dot{R}, \dot{x}_n, \dot{\xi} = u$, as generalized velocities. The Lagrangian for this problem is given by $L = T - U$, and equations of motion are obtained by applying Lagrange's equations to this Lagrangian.

Lagrange's equations of motion are, in general,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial U}{\partial q_i} = 0, \tag{21}$$

where q_i represents any of R, x_n or ξ. Of particular importance is the fact that the co-ordinate ξ does not appear explicitly in the Lagrangian. Consequently, the equation for q_1 gives $d/dt(\partial T/\partial u) = 0$, and the translational momentum, defined as

$$G \equiv \partial T/\partial u \tag{22}$$

is a constant of the motion. G is given by means of (11) as

$$G = M_{10}\dot{R}+M_{11}u+\sum_{2}^{\infty}M_{1n}\dot{x}_n = \sum_{0}^{\infty}M_{1i}\dot{q}_i. \tag{23}$$

With (11), the equations of motion are given in general by

$$\sum_j M_{ij}\ddot{q}_j+\sum_k\sum_j\left[\frac{\partial M_{ij}}{\partial q_k}-\frac{1}{2}\frac{\partial M_{kj}}{\partial q_i}\right]\dot{q}_k\dot{q}_j+\frac{\partial U}{\partial q_i} = 0. \tag{24}$$

3. The equilibrium state of a translational cavity

Solutions of the equations of motion will now be sought for the case of a cavity of constant size and shape, moving with a constant translational velocity. The constant values of the co-ordinates and translational velocity will be written

R_e, x_n and u_e. Such an equilibrium situation is specified by solutions of (24) with R, u and x_n held constant. Using (18), (14), (15) and (19), we find the equilibrium equations for R_e

$$\frac{\rho m u_e^2}{4} = \frac{2\sigma s}{R_e} + (P_0 - p_{ge})\, w, \tag{25}$$

and for x_n

$$\frac{\rho u_e^2}{4}\frac{\partial m}{\partial x_n} = \frac{3\sigma}{R_e}\frac{\partial s}{\partial x_n} + (P_0 - p_{ge})\frac{\partial w}{\partial x_n}. \tag{26}$$

It is noted here that R_e and p_{ge} are the equilibrium values of the radius and gas pressure of a translating, deformed cavity. They differ from the values of R and p_g that would result if the same cavity were at rest. The momentum of the equilibrium cavity is given by (23) and (15) as

$$G = M_{11} u_e = \tfrac{2}{3}\pi \rho m R_e^3 u_e. \tag{27}$$

If (25) is used to eliminate $(P_0 - p_{ge})$ from (26), one obtains

$$\beta\left(\frac{\partial m}{\partial x_n} - \frac{m}{w}\frac{\partial w}{\partial x_n}\right) = 12\frac{\partial s}{\partial x_n} - 8\frac{s}{w}\frac{\partial w}{\partial x_n}. \tag{28}$$

The parameter β, given by

$$\beta = \rho u_e^2 R_e/\sigma \tag{29}$$

is known as the Weber number.

The equilibrium values of x_n are determined by (28), which defines a family of equilibrium cavity shapes in terms of the Weber number β. Once β is specified and the equilibrium shape is known, the equilibrium radius is related to the translational velocity and the gas pressure by (25) and (29). The radius R is the coefficient of the zeroth Legendre polynomial P_0 in the expansion of the position of the cavity surface, (1). Thus, the equilibrium radius is a kind of average radius of the equilibrium cavity.

Equation (25) indicates that the effect of translational motion is to contribute a dynamic pressure equal to $\rho m u_e^2/4w$ which opposes a positive hydrostatic pressure and the pressure due to surface tension, and which tends to enlarge the cavity. Two principal objectives of this study of the equilibrium cavity are, first, to find the equilibrium values of the shape co-ordinates x_n, and second, to examine the effects of the dynamic pressure on the equilibrium radius of the cavity.

The equilibrium shape

Equilibrium values of x_n, as expansions in powers of the Weber number β, have been found from (28) by means of an iteration procedure. The quantities m, s and w appearing in this equation are given as functions of x_n by (17), (20) and (16). The zero-order term in the expansion of x_n is found by considering a stationary cavity. In this case u_e and β are both zero, and the left-hand side of (28) vanishes. The zero-order values of x_n are then given by roots of the zero-order equations

$$\tfrac{3}{2}\,\partial s/\partial x_n - (s/w)\,\partial w/\partial x_n = 0.$$

We see from (20) and (16) that s and w contain no terms linear in x_n. As a result, the zero-order equations have a solution $x_n = 0$. This solution corresponds to the well-known result that the equilibrium shape of a stationary cavity is a sphere.

A first-order approximation is obtained by keeping terms of order β in (28) and then solving this set of equations for each x_n. Because each x_n is at least of order β, products of the form $x_n\beta$ and $x_n x_k$ are at least of order β^2 and are omitted in this approximation. For example, to first order, equation (28) for $n = 2$ is

$$\beta(-9/5) = 12(8/5)x_2 - 8(6/5)x_2.$$

It is found that, to first order in β, the equilibrium shape is described by $x_2 = -(3/16)\beta$, and $x_3 = x_4 = \ldots = 0$; this shape is an oblate spheroid.

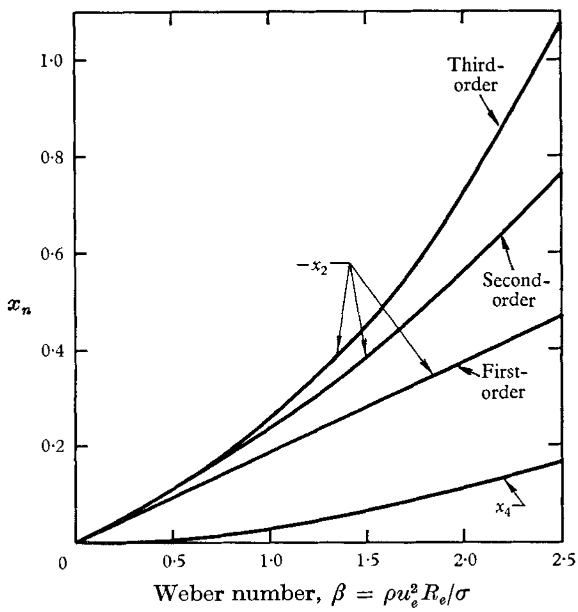

FIGURE 2. Equilibrium values of x_2 and x_4 as a function of the Weber number.

Higher-order approximations are obtained in a similar way. The equilibrium values of x_n that have been obtained in this manner are

$$\left.\begin{aligned} x_2 &= -0{\cdot}1875\beta - 0{\cdot}04721\beta^2 - 0{\cdot}01951\beta^3 + \ldots, \\ x_4 &= 0{\cdot}02612\beta^2 + \ldots, \\ x_6, x_8, x_{10}, \ldots &= O(\beta^3) \text{ or higher order}, \\ x_3 &= x_5 = x_7 = \ldots = 0. \end{aligned}\right\} \qquad (30)$$

The equilibrium values of the mass, area and volume ratios, calculated from (17), (20), (16) and (30), are

$$\left.\begin{aligned} m &= 1{\cdot}0000 + 0{\cdot}3375\beta + 0{\cdot}1513\beta^2 + 0{\cdot}08096\beta^3 + \ldots, \\ s &= 1{\cdot}0000 + 0{\cdot}02813\beta^2 + 0{\cdot}01416\beta^3 + 0{\cdot}008153\beta^4 + \ldots, \\ w &= 1{\cdot}0000 + 0{\cdot}02109\beta^2 + 0{\cdot}01025\beta^3 + 0{\cdot}005828\beta^4 + \ldots. \end{aligned}\right\} \qquad (31)$$

The higher-order terms in these approximations for x_n, m, s and w decrease rapidly when β is small. The successive approximations for x_2 and the second-order approximation for x_4 are plotted in figure 2. The value of x_2 is seen to grow rapidly when β is greater than about 2. We see from the graph that the power series for x_2 appears to be most useful when β is less than about 1·2 or 1·3.

Thus, for small Weber numbers, one can calculate the equilibrium values of x_n, which determine the equilibrium cavity shape, and the equilibrium values of the shape-dependent quantities m, s and w. The equilibrium shape is described by the function f, defined in (1) as $f = 1+\Sigma x_n P_n(\cos\theta)$. By means of (30) the shape function f is given to second order in β by

$$f = 1-(0{\cdot}1875\beta+0{\cdot}04721\beta^2)P_2+0{\cdot}02612\beta^2P_4. \tag{32}$$

The equilibrium shape described by this equation is plotted in figure 3 for a Weber number of $\beta = 1{\cdot}20$. (The numbers appearing on this graph refer to the pressure distribution at the cavity surface and will be explained later.) The equilibrium shape is seen to have front and back symmetry, that is, symmetry about a plane passing through the centre of the cavity, perpendicular to the direction of translation. This symmetry results from the fact that all odd co-ordinates $x_3, x_5, \ldots$ are zero.

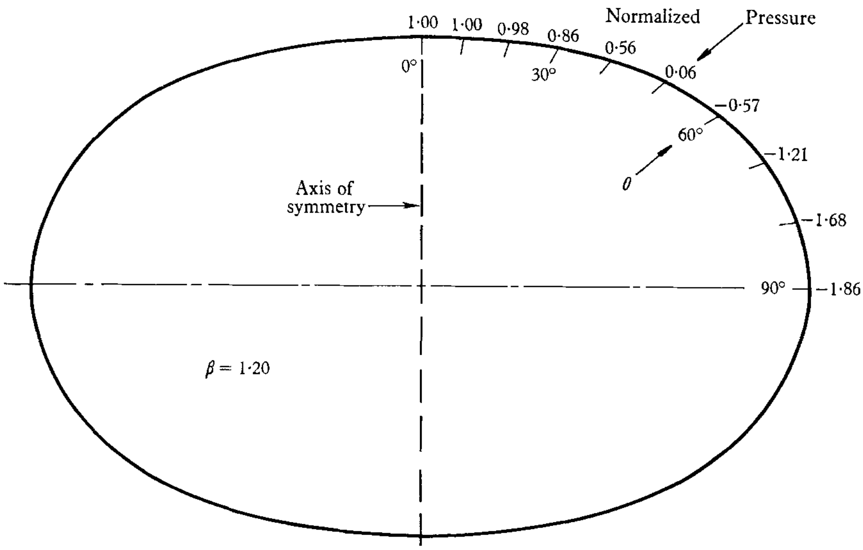

FIGURE 3. Equilibrium shape and pressure distribution at the surface of a translating cavity with a Weber number $\beta = 1{\cdot}20$.

The equilibrium radius

Once the cavity shape is known as a function of β, the equilibrium radius can be obtained from (25). Let us assume that the cavity contains n moles of an ideal gas under isothermal conditions. The equilibrium gas pressure is then given by $p_{ge} = nBT/V = 3nBT/4\pi R_e^3 w$, where B is the universal gas constant and T is the absolute temperature. Substituting this result and (29) into (25), one finds that the equilibrium radius corresponding to given values of n and β is given by solutions of the equation

$$\frac{m\beta\sigma}{4R_e}+\frac{3nBT}{4\pi R_e^3} = \frac{2\sigma s}{R_e}+P_0 w. \tag{33}$$

The terms on the left represent an outward pressure that tends to enlarge the cavity. As a result, the equilibrium radius of a translating cavity is greater than the equilibrium radius of the same cavity at rest.

For example, let us consider a stationary, spherical cavity in water with $P_0 = 0$ and with an equilibrium radius of $5{\cdot}00\mu$. If this cavity is given translational motion so that its Weber number becomes $\beta = 1{\cdot}00$, then its equilibrium radius is increased to $5{\cdot}42\mu$, or $1{\cdot}084$ times its original value. The equilibrium volume is increased to $1{\cdot}32$ times its original value, and the velocity corresponding to $\beta = 1{\cdot}00$ and $R_e = 5{\cdot}42\mu$ is $u_e = 368\,\text{cm/sec}$.

A translating cavity can have an equilibrium radius even if it is empty of gas. If a quantity α is defined as

$$\alpha = P_0 R_e / 2\sigma, \tag{34}$$

then (33) may be written for the empty cavity case, $n = 0$, as

$$\begin{aligned} \alpha &= (m\beta/8w) - (s/w) \\ &= -1{\cdot}0000 + 0{\cdot}1250\beta + 0{\cdot}03516\beta^2 + 0{\cdot}01258\beta^3 + 0{\cdot}005774\beta^4 + \dots . \end{aligned} \tag{35}$$

This equation relates values of α and β that correspond to an empty cavity. The value of β lies within the useful range of this power series when P_0 is negative. For example, if α equals $-0{\cdot}77$, then the corresponding empty cavity Weber number is $1{\cdot}20$. In other words, the cavity whose shape is shown in figure 3 could be empty of gas if the hydrostatic pressure were negative and related to R_e by $P_0 R_e/2\sigma = -0{\cdot}77$.

Two dimensionless quantities introduced in this chapter are the Weber number $\beta = \rho u_e^2 R_e/\sigma$ and the quantity $\alpha = P_0 R_e/2\sigma$. The Weber number is proportional to the ratio of the inertial pressure $\frac{1}{2}\rho u_e^2$ and the pressure $2\sigma/R_e$ related to surface tension. The quantity α is the ratio of the hydrostatic pressure P_0 and the pressure related to surface tension.

Pressure distribution at the surface

The pressure in the liquid about an equilibrium cavity may be calculated from the Bernoulli equation

$$p = P_0 - \tfrac{1}{2}\rho v^2 + \rho\, \partial\phi/\partial t.$$

The partial derivative $\partial\phi/\partial t$ is taken at a fixed point in space. The value of ϕ at a point fixed with respect to the moving cavity is constant for the equilibrium case, and, therefore, the partial time derivative at a point moving with the cavity, written $(\partial\phi/\partial t)_c$, is zero. Because the cavity moves in the z-direction with velocity u, these two partial derivatives are related by

$$(\partial\phi/\partial t)_c = 0 = (\partial\phi/\partial t) + u(\partial\phi/\partial z).$$

From this last expression one obtains $\partial\phi/\partial t = -u(\partial\phi/\partial z)$. Therefore the pressure in the liquid is given by

$$(p - P_0)/\rho = -\tfrac{1}{2}v^2 - u\,\partial\phi/\partial z.$$

The values of v^2 and $\partial\phi/\partial z$ may be calculated from (2) along with the values of b_{n1} and the equilibrium values of x_n. In this manner the pressure distribution in the liquid at the cavity surface has been calculated for the equilibrium case through first order in β. The result may be written as

$$\frac{p_s - P_0}{\frac{1}{2}\rho u_e^2} = -\tfrac{1}{2} + \tfrac{3}{2}(1 + \tfrac{27}{70}\beta) P_2 - \tfrac{81}{140}\beta P_4 + \ldots, \tag{36}$$

where $p_s(\theta)$ is the pressure in the liquid at the cavity surface. The pressure distribution at the surface has been calculated from (36) for a Weber number $\beta = 1{\cdot}20$, and the values of the normalized pressure $(p_s - P_0)/\frac{1}{2}\rho u_e^2$ are indicated as a function of angle in figure 3.

4. Stability of the equilibrium state

The stability of the equilibrium size and shape will be examined by considering the energy involved in displacements of R and x_n from their equilibrium values. The displacements of R and x_n will be made so that the momentum G, which is a constant of the motion, is held fixed. First, it is convenient to transform the equations of motion, (21). These equations, in their present form, are written in terms of functions of the generalized co-ordinates R and x_n and the translational velocity u. It is possible to rewrite these equations in terms of R and x_n and the constant momentum G, rather than u.

This transformation is effected by a method known as Routh's procedure (Goldstein 1950, p. 219). A function called the Routhian is defined as

$$\mathscr{R} = L - Gu. \tag{37}$$

The equations of motion for R and x_n are then given by Lagrange's equations, but with the Routhian playing the part of the Lagrangian,

$$\frac{d}{dt}\left(\frac{\partial \mathscr{R}}{\partial \dot{R}}\right) - \frac{\partial \mathscr{R}}{\partial R} = 0; \quad \frac{d}{dt}\left(\frac{\partial \mathscr{R}}{\partial \dot{x}_n}\right) - \frac{\partial \mathscr{R}}{\partial x_n} = 0. \tag{38}$$

It is an essential part of this method that $\mathscr{R}$ be written explicitly as a function of the constant momentum G, rather than u.

Using $L = T - U$, equations (11), (18) and (37), and recalling that $\dot{q}_1$ is the same as u, one obtains

$$\mathscr{R} = \tfrac{1}{2}\sum_i{}' \sum_j{}' M'_{ij}\dot{q}_i\dot{q}_j + \frac{G}{M_{11}} \sum_j{}' M_{1j}\dot{q}_j - \Phi, \tag{39}$$

where
$$M'_{ij} = M_{ij} - \frac{M_{1i}M_{1j}}{M_{11}}, \tag{40}$$

$$\Phi = \frac{G^2}{2M_{11}} + U, \tag{41}$$

and where the symbol Σ_i' indicates that the summation does not include $i = 1$. Equation (39) gives the Routhian as a function of the generalized co-ordinates $q_i\,(i \neq 1)$ and generalized velocities $\dot{q}_i\,(i \neq 1)$ and the translational momentum G. By analogy with the role played by U in the Lagrangian, the quantity Φ is interpreted as an effective potential energy for the translating cavity.

The total energy of the cavity $E = T+U$ may also be transformed to give

$$E = \tfrac{1}{2}\sum_i{}' \sum_j{}' M'_{ij}\dot{q}_i\dot{q}_j + \Phi. \tag{42}$$

The first term of E is the kinetic energy associated with change of R and x_n. The second term is the effective potential energy. Here it is again seen that the kinetic energy of translation contributes to the effective potential energy.

Equations for the equilibrium situation are obtained from (38) and (39) with R and x_n held constant and are

$$\partial\Phi/\partial R = 0; \quad \partial\Phi/\partial x_n = 0. \tag{43}$$

Taking the indicated derivatives of Φ, and replacing G by its equivalent at equilibrium, $M_{11}u_e$, one easily verifies that these equilibrium equations are identical to those obtained from the original Lagrangian. The equilibrium equations determine an extremum of the potential Φ. The equilibrium will be stable if this extremum is a minimum. Since Φ is the total energy of the equilibrium state, a stable equilibrium corresponds to a minimum of the total energy.

Eigenvalues of the effective potential energy

The effective potential energy may be expanded about its equilibrium value as

$$\Phi = \Phi_e + \tfrac{1}{2}\sum_i{}' \sum_j{}' (\partial^2\Phi/\partial q_i\partial q_j)_e\,\delta q_i\delta q_j + \dots, \tag{44}$$

where the partial derivatives are to be evaluated for the equilibrium configuration. The quadratic form appearing in (44) represents the effective potential energy change for any displacement from the equilibrium configuration. The value of Φ at equilibrium will be a minimum, and the equilibrium stable, if this quadratic form is positive for all such displacements. The quadratic form can be represented by the symmetric matrix whose elements are

$$\Phi_{ij} = (\partial^2\Phi/\partial q_i\partial q_j)_e. \tag{45}$$

The eigenvalues λ_k of this matrix are real because the matrix is symmetric. If the corresponding eigenvectors are denoted $\delta q'_k$, then the quadratic form can be written in the diagonal representation as $\Sigma_k\lambda_k(\delta q'_k)^2$. It is seen from this expression that the quadratic form is positive and the equilibrium stable if all eigenvalues λ_k are positive. Our goal is to obtain these eigenvalues. First, however, we must calculate the matrix elements Φ_{ij}. One is reminded here that the subscripts i, j do not assume the value one.

If the gas in the cavity behaves as an ideal gas under isothermal conditions, then Φ is given by (41), (14), (15), (18) and (19) as

$$\Phi = \frac{3G^2}{4\pi\rho mR^3} + 4\pi R^2\sigma s + \frac{4\pi}{3}R^3P_0w - c\ln R^3w, \tag{46}$$

where $c = p_gV$ is a constant. The momentum and gas content of the cavity are held fixed. The elements Φ_{ij} are obtained by taking second derivatives of Φ and evaluating them for the equilibrium state determined by (43). For this purpose one needs the equilibrium values of x_n, m, s and w, given by (30) and (31). The calculations have been extended to the order in β sufficient for later calculations. The first two diagonal terms are given to first order in β by

$$\Phi_{00} = 4\pi\sigma(4+6\alpha+\tfrac{3}{4}\beta),$$
$$\Phi_{22} = 4\pi\sigma R_e^2(0\cdot8000+0\cdot3600\beta).$$

The third diagonal will be used to second order, and is

$$\Phi_{33} = 4\pi\sigma R_e^2(1\cdot429-0\cdot2755\beta-0\cdot06417\beta^2).$$

The remaining diagonal elements are given to zero order by

$$\Phi_{nn}(n \geqslant 2) = 4\pi\sigma R_e^2(n+2)(n-1)/(2n+1).$$

The off-diagonal elements are, to first order in β,

$$\Phi_{02} = -\tfrac{3}{5}\pi\sigma R_e(7+3\alpha)\beta,$$
$$\Phi_{n-1\,n+1}(n \geqslant 3) = \tfrac{9}{2}\pi\sigma R_e^2[n(n+1)/(2n-1)(2n+3)]\beta.$$

All other off-diagonal elements are of second or higher order in β.

It is noted that all off-diagonal elements are at least of order β, while the diagonal elements consist of terms of zero order as well as terms of order β and higher. This fact allows one to calculate the eigenvalues of the matrix Φ_{ij} as a series of terms of increasing order in β. For small values of β, the eigenvalues may be approximated by the first few terms of these series. The following notation will be adopted: Φ_{ij} is written

$$\Phi_{ij} = \Phi_{ij}^{(0)}+\Phi_{ij}^{(1)}+\Phi_{ij}^{(2)}+\dots,$$

where $\Phi_{ij}^{(n)}$ is the contribution to Φ_{ij} of order β^n.

The eigenvalues are given by solutions λ_k of the equation $\det(\Phi_{ij}-\lambda\delta_{ij}) = 0$, where δ_{ij} is the Kronecker delta. It can be shown that the eigenvalues are given to second order in β by

$$\lambda_k = \Phi_{kk}^{(0)}+\Phi_{kk}^{(1)}+\Phi_{kk}^{(2)}+\sum_{i\neq k}{}' \frac{(\Phi_{ik}^{(1)})^2}{\Phi_{kk}^{(0)}-\Phi_{ii}^{(0)}}. \tag{47}$$

Eigenvalues have been calculated from (47) through the order in β needed in the following section. The first two eigenvalues are given to first order by

$$\lambda_0 = 16\pi\sigma(1+3\alpha/2+3\beta/16),$$
$$\lambda_2 = 4\pi\sigma R_e^2(0\cdot8000+0\cdot3600\beta).$$

The third eigenvalue is given to second order by

$$\lambda_3 = 4\pi\sigma R_e^2(1\cdot429-0\cdot2755\beta-0\cdot1406\beta^2).$$

The higher eigenvalues are given to zero order by

$$\lambda_n(n \geqslant 2) = 4\pi\sigma R_e^2(n+2)(n-1)/(2n+1).$$

The equilibrium state will be stable when all of these eigenvalues are positive. If the quantity $\alpha = P_0R_e/2\sigma$ is positive, then all eigenvalues are positive, and the equilibrium is stable for small values of β. A principal conclusion, therefore, is that stable solutions for the equilibrium cavity size and shape do exist for a range of Weber numbers near the value zero.

Instabilities of the equilibrium state

The equilibrium will be unstable if any eigenvalue is negative. Two such instabilities have been found for this problem. One form of instability occurs when the hydrostatic pressure is negative. Then, when the equilibrium radius exceeds a critical value, the radius will grow without bound; therefore, this instability is one that involves the size of the cavity rather than its shape. The critical radius at which the instability occurs is a function of the negative hydrostatic pressure. Such an instability is well known for the case of non-translating gas bubbles, and will occur when the negative pressure is related to the equilibrium radius by $|P_0| \geqslant 4\sigma/3R_e$. The instability occurs for translating cavities when λ_0 equals zero, or

$$\alpha = -\tfrac{2}{3}-\tfrac{1}{8}\beta. \tag{48}$$

From the definitions of α and β, equations (34) and (29), the condition for this instability becomes

$$P_0 < 0; \quad |P_0| \geqslant (4\sigma/3R_e)+(\rho u_e^2/4).$$

A second instability predicted by these results will occur when λ_3 becomes zero. The second-order expression for λ_3 becomes zero when β equals 2·34. Therefore, according to this approximation, the equilibrium cavity is unstable when the Weber number exceeds 2·34. Because α does not appear in the expression for λ_3, this instability is independent of the hydrostatic pressure. The displacement corresponding to the eigenvalue λ_3 is given by the third eigenvector, which consists primarily of a displacement δx_3. Therefore, this instability is associated with a change of x_3 and involves the shape of the cavity, rather than its size. No other instabilities have been found within this range of values of β. Values of λ_2 and λ_3 are shown in figure 4.

The range of stability

Finally, let us consider the range of parameters for which stable equilibrium states exist. The values of R_e and u_e for which β equals the approximate stability limit 2·34 are shown in figure 5. This line is calculated from the definition $\beta = \rho u_e^2 R_e/\sigma$, using $\beta = 2{\cdot}34$ and the values for water, $\sigma = 73\,\text{dynes/cm}^2$ and $\rho = 1{\cdot}0\,\text{g/cm}^3$. All equilibrium states represented by points lying below this line have a stable shape.

The values of α and β that correspond to equilibrium cavities are shown in figure 6. All points above curve (1) represent equilibrium cavities that contain gas. A point on curve (1) represents the limiting case of an equilibrium, empty cavity. Curve (1) is calculated from equation (35). Points below curve (1) do not represent equilibrium states, for they would correspond to a negative amount of gas in the cavity. The equilibrium states are stable when α and β are within two stability limits. The limit for size stability is given by (48) and is indicated by curve (2) in figure 6. The limit for shape stability is given by $\beta = 2{\cdot}34$ and is indicated by curve (3). The stability limits divide the range of equilibrium solutions in figure 6 into four regions. Points in region I represent stable equilibrium solutions. Equilibrium cavities represented by points in region II have an unstable size and will grow without bound. Points in region III represent equilibrium cavities that are unstable in shape. There are no equilibrium states in region IV.

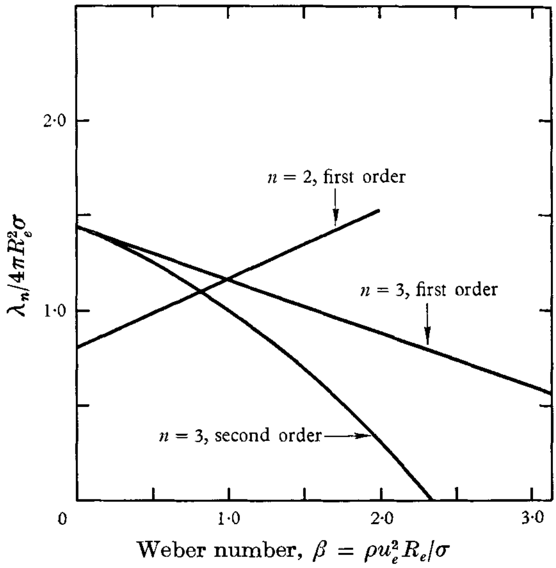

FIGURE 4. Eigenvalues λ_2 and λ_3 of the effective potential energy as a function of the Weber number.

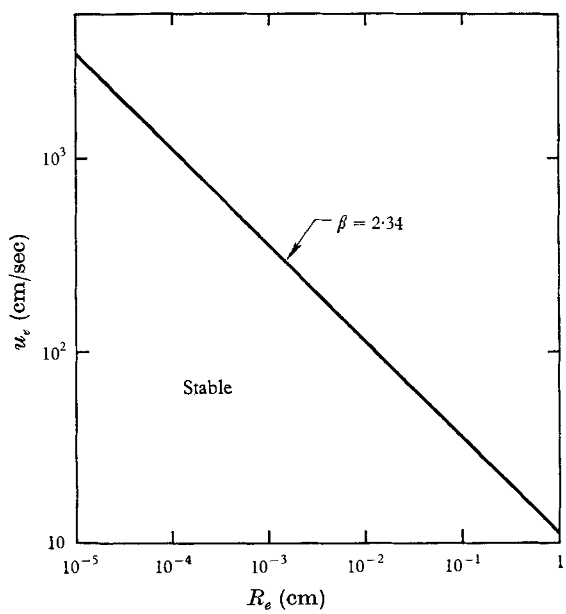

FIGURE 5. The range of u_e and R_e for shape stability.

It is seen from figure 6 that stable empty cavities exist between the intersections of curve (1) with curves (2) and (3), at the points $\alpha = -0{\cdot}80$, $\beta = 1{\cdot}08$ and $\alpha = -0{\cdot}17$, $\beta = 2{\cdot}34$. For a given negative hydrostatic pressure the possible stable equilibrium radii of empty cavities lie between $R_{\min} = -0{\cdot}17(2\sigma/P_0)$ and $R_{\max} = -0{\cdot}80(2\sigma/P_0)$. The smaller radius corresponds to $\beta = 2{\cdot}34$, the limit for shape stability; the larger radius corresponds to $\beta = 1{\cdot}08$, the limit for size stability. For example, if P_0 is $-0{\cdot}1$ bar, stable empty cavity equilibrium radii lie between $2{\cdot}5$ and $12{\cdot}0\mu$. These results for translating empty cavities differ greatly from the properties of stationary empty cavities, which have an equilibrium radius when P_0 is negative. This equilibrium radius is given by $R_e = -2\sigma/P_0$ and is always unstable. There is no stable empty cavity equilibrium for a stationary cavity.

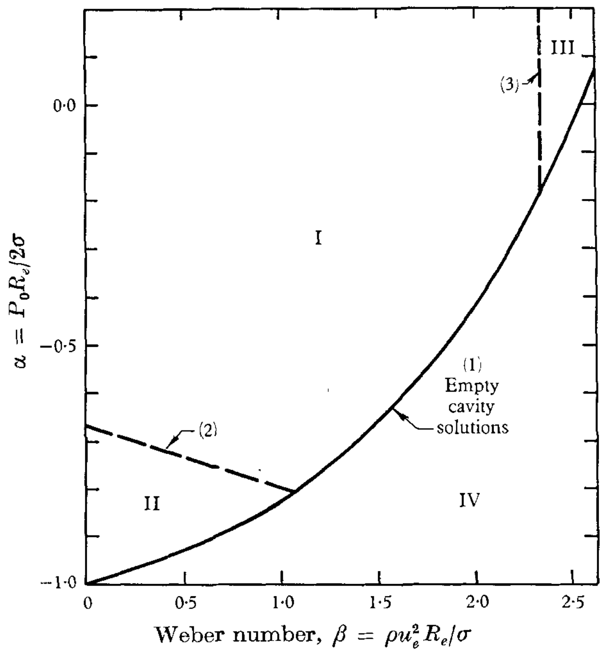

FIGURE 6. The range of α and β for stable and unstable equilibrium cavities. I, stable equilibrium; II, unstable size; III, unstable shape; IV, no equilibrium states.

5. Conclusions

A principal conclusion of this study is that translating cavities have an equilibrium size and shape for a range of values of the Weber number β. The equilibrium shape is approximately an oblate spheroid, and the equilibrium radius is greater than it would be if the cavity were stationary. This increase in radius occurs because the average pressure in the liquid surrounding a translating cavity is reduced.

A second conclusion is that the equilibrium states are stable for a range of values of α and β. Beyond this range an equilibrium state may become unstable in either of two ways.

(i) The equilibrium shape becomes unstable when the Weber number exceeds the approximate value 2·34. This limiting value was obtained by extrapolation of a second-order expression in β and, therefore, is not expected to be highly accurate. The range of velocities at which the cavity shape is stable is quite large when the equilibrium radius is small. For example, according to figure 5, a cavity with a radius of one micron will have a stable shape for velocities up to about 1300 cm/sec.

(ii) The equilibrium radius is unstable when the approximate inequality $\alpha \leqslant -\frac{2}{3} - \frac{1}{8}\beta$ is satisfied. This instability occurs only when P_0 is negative. Empty cavities may lie within the stable range of α and β if P_0 is negative.

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